

# Eigenstates of the Atom-Field Interaction and the Binding of Light in Photonic Crystals

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We solve for the exact atom-field eigenstates of a single atom in a three dimensional spherical cavity, by mapping the problem onto the anisotropic Kondo model. The spectrum has a rich bound state structure in comparison with models where the rotating wave approximation is made. It is shown how to obtain the Jaynes-Cummings model states in the limit of weak coupling. Non-perturbative Lamb shifts and decay rates are computed. The massive Kondo model is introduced to model light localization in the form of photon-atom bound states in photonic crystals.

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## 1. Introduction

The system of two-level atoms coupled to radiation is a fundamental problem of physics, and much work has been devoted to its study, especially in the context of quantum optics[1]. For the most part, a satisfactory understanding of the problem is obtained after making some approximations, most importantly the rotating-wave and slowly varying envelope approximations. Though these approximations are well-justified near resonance, it is important to have an understanding of what is missed in making these approximations, and to ascertain whether there are novel effects of any significance.

In this paper we present an exact solution to the problem of a single two-level atom in a spherical cavity, without making the above approximations. This is a considerably more complicated problem because of the vacuum fluctuations. The solution is accomplished by mapping the optical problem onto the Kondo problem of massless fermions interacting with a magnetic impurity, as was done for a one-dimensional fiber geometry in [2].

The novel features of our solution, which are unanticipated based on previous studies, have to do with the spectrum of eigenstates of the atom-field interaction. For a single cavity mode in the rotating-wave approximation, the atom-field eigenstates are known from the solution of the Jaynes-Cummings model. The parameters of the latter model are the two-level splitting  $\omega_0$ , and the coupling  $\alpha$  of the atom to the radiation. The latter determines the decay rate  $\Gamma_{\text{decay}} = \alpha^2 L$  where  $L$  is the volume. The eigenstates of the Jaynes-Cummings model correspond to doublets of atomically dressed superpositions of  $N$  and  $N - 1$  photon states. Our solution is given below in terms of a scattering theory description for a spectrum of massless particles, and we describe how information about the atom-field eigenstates is encoded in the scattering matrices. As a result, we find that the spectrum of states in our model is much richer, in a way that depends on the dimensionless ratio  $g = \Gamma_{\text{decay}}/\pi\omega_0$ , which is a quantum coupling. Roughly speaking, we find that the one-photon ( $N = 1$ ) eigenstates are  $1/g$  in number, the lightest becoming identified with the Jaynes-Cummings one at weak coupling (small  $g$ ) and near resonance, and the higher ones arising as bound states, and as solitons. Interestingly, at the special strong coupling point  $g = 1/2$ , the fundamental Jaynes-Cummings eigenstate becomes unbound and disappears entirely from the spectrum, leaving only solitonic states. In addition to these spectral features we are able to compute non-perturbative expressions for the Lamb-shifted energy splittings and decay widths.

The above spectrum has a sine-Gordon-like character. Sine-Gordon theory is not unfamiliar in this context, as it is well-known that a resonant dielectric *medium* of two-level

atoms in a semi-classical limit is well-described by the classical sine-Gordon theory. This is the phenomena of self-induced transparency[3]. Our work can be interpreted as showing that the propagating spectrum in the theory of self-induced transparency has an important significance for a *single* atom in a cavity interacting with fully quantized radiation. Our theory thus provides a bridge between the theory of self-induced transparency and the Jaynes-Cummings model.

In the final sections of the paper we apply our techniques to the problem of an atomic impurity in a medium with a photonic bandgap, with applications to photonic crystals in mind[4]. Photonic crystals, envisaged in early works of Yablonovich and John[5][6], are periodic dielectric structures exhibiting gaps in the allowed energies of photon propagation, in close analogy to electronic band structure. Important effects are expected when these materials are doped with atomic impurities. Specifically, the spontaneous emission of an atom is severely inhibited if the energy splitting of levels in the atom coincides with an energy that is forbidden to propagate in the material. Under these circumstances, the light can form a bound state with the atom, this photon-atom bound state being the optical analog of an electron-impurity-level bound state in the gap of a semiconductor[7]. In this paper we introduce a toy model where these effects can be studied exactly by adding a mass term to the Kondo model in a way that preserves the integrability of the latter. Here, the origin of the photonic bandgap is the gap between polariton branches. We solve this massive Kondo model at a special point where it is equivalent to a free fermion theory, and indeed find a photon-atom bound state with the expected properties. We can also compute exactly the binding energy of this state. A different integrable model of this phenomenon was studied in [8] wherein the rotating wave approximation is made.

We present our results in the following way. In section 2 we use spherical symmetry to reduce the problem from three spatial dimensions to one, and in section 3 describe the resulting theory as a boundary quantum field theory. For the resulting theory to be exactly solvable, one needs to make an approximation that favors photons in the vicinity of the resonance; such an approximation is not necessary in the one-dimensional case. In section 4 we map the problem onto the anisotropic Kondo model, describe the scattering spectrum in infinite volume, and present exact reflection S-matrices expressed in terms of the Lamb-shifted resonant energies and decay widths. In section 5 we describe the meaning of topological charge in the optical context. In section 6 we describe how information about the atom-field eigenstates is encoded in the reflection S-matrices by computing the one-particle finite volume spectrum and comparing this to the Jaynes-Cummings spectrum.

Here we show how for each particle of the infinite volume spectrum one can associate polariton-like states with optical phonon behavior at large and small energies. In section 7 the massive Kondo model is introduced to model atomic impurities in photonic crystals. The latter model is solved at the special point  $g = 1/2$  in section 8, and this solution is used in section 9 to study the localization of light.

## 2. Spherical Dimensional Reduction

We consider a two-level ‘atom’, with spherically symmetric states  $|\pm\rangle$  which are eigenstates of the unperturbed atomic hamiltonian  $H_0^{\text{atom}}$ :

$$H_0^{\text{atom}} |\pm\rangle = \pm \frac{\omega_0}{2} |\pm\rangle, \quad (2.1)$$

such that the energy splitting is  $\omega_0$ . We work in spherical coordinates  $r, \theta, \phi$ , and place the atom at  $r = 0$ . We will focus on electric-dipole transitions, thus we further assume  $|-\rangle$  is an angular momentum eigenstate  $(l, m = 0)$ , whereas  $|+\rangle$  is an  $(l + 1, m = 0)$  state. The wave-functions of these states are then

$$\langle \vec{r} | + \rangle = f_+(r) Y_{l+1,0}(\theta, \phi), \quad \langle \vec{r} | - \rangle = f_-(r) Y_{l0}(\theta, \phi). \quad (2.2)$$

In coupling the atom to radiation, one uses the electric-multipole expansion as described in e.g. [9]. Expanding electric and magnetic fields in frequency components,

$$\vec{E}(\vec{r}, t) = \int d\omega e^{-i\omega t} \vec{E}_\omega(\vec{r}) + e^{i\omega t} \vec{E}_\omega^\dagger(\vec{r}), \quad (2.3)$$

and similarly for  $\vec{B}$ , and keeping only the electric-dipole term in the multipole expansion, one has

$$\begin{aligned} \vec{B}_\omega(\vec{r}) &= 2\omega^{3/2} a(\omega) j_1(\omega r) \vec{X}_{10}(\theta, \phi) \\ \vec{E}_\omega(\vec{r}) &= 2i\omega^{1/2} a(\omega) \vec{\nabla} \times j_1(\omega r) \vec{X}_{10}(\theta, \phi) \end{aligned} \quad (2.4)$$

where  $\vec{X}_{lm} = \vec{L} Y_{lm} / \sqrt{l(l+1)}$ , and  $j_l$  is a spherical Bessel function. Using orthogonality relations for  $\vec{X}_{lm}$  and  $j_l$ , the free field hamiltonian is

$$H_0^{\text{field}} = \frac{1}{8\pi} \int d^3\vec{r} (\vec{E}^2 + \vec{B}^2) = \int_0^\infty d\omega \omega a^\dagger(\omega) a(\omega). \quad (2.5)$$

In the quantum theory one has the commutation relation

$$[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega'). \quad (2.6)$$

The atom is coupled to the radiation with the usual dipole interaction  $H_{int} = -\vec{d} \cdot \vec{E}$  where  $\vec{d}$  is the electric dipole operator of the atomic electron,  $\vec{d} = e\vec{r}$ . Letting  $\vec{r} = r\hat{r}$ , one can show

$$\hat{r} \cdot \vec{E}_\omega(\vec{r}) = -\sqrt{8\omega} a(\omega) \frac{j_1(\omega r)}{r} Y_{10}. \quad (2.7)$$

We assume that the wavelength of relevant photons is large compared to atomic dimensions, such that the  $r$  dependence in (2.7) can be pulled out of the matrix element:

$$-\langle +|\vec{d} \cdot \vec{E}_\omega(\vec{r})|-\rangle \approx \lim_{r \rightarrow 0} \left( \frac{j_1(\omega r)}{r} \right) \sqrt{8\omega} a(\omega) \langle +|erY_{10}|-\rangle. \quad (2.8)$$

The matrix element  $\langle +|erY_{10}|-\rangle$  can be related to the usual reduced dipole moment  $d$  defined as  $\langle +|d_m|-\rangle = d\delta_{m0}$ , where  $d_m$  is the  $m = \pm 1, 0$  spherical component of the vector  $\vec{d}$ . Using  $\vec{r}_m = r\sqrt{\frac{4\pi}{3}}Y_{1m}$ , one finds

$$\langle +|erY_{10}|-\rangle = \sqrt{\frac{3}{4\pi}}d, \quad (2.9)$$

where

$$d = e \frac{(l+1)}{\sqrt{(2l+3)(2l+1)}} \int_0^\infty dr r^3 f_+^*(r) f_-(r). \quad (2.10)$$

Finally, using  $\lim_{x \rightarrow 0} j_1(x)/x = 1/3$ , and the reality of  $Y_{l0}$ , the complete hamiltonian is

$$H = H_0^{\text{field}} + H_0^{\text{atom}} + H_{int} \quad (2.11)$$

where

$$H_0^{\text{atom}} = \frac{\omega_0}{2} S_3, \quad (2.12)$$

and

$$H_{int} = \sqrt{\frac{2}{3\pi}} \left\{ \int_0^\infty d\omega \omega^{3/2} (a(\omega)e^{-i\omega t} + a^\dagger(\omega)e^{i\omega t}) \right\} (d S^+ + d^* S^-). \quad (2.13)$$

Here,  $S_3, S^\pm$  are Pauli matrix operators:

$$[S_3, S^\pm] = \pm 2S^\pm, \quad [S^+, S^-] = S_3. \quad (2.14)$$

### 3. Boundary Field Theory Description

In this section we will give a boundary field theory description of the model of the last section. Boundary field theories live on the half-line in space,  $x \geq 0$ , with non-trivial

interactions at the boundary. The half-line occurs automatically in the spherical reduction since  $r \geq 0$ , and the boundary interaction is at  $r = 0$ .

The free hamiltonian  $H_0^{\text{field}}$  is equivalent to a free boson on the half-line:

$$H_0^{\text{field}} = \int_0^\infty dr \frac{1}{2} \left( (\partial_t \phi)^2 + (\partial_r \phi)^2 \right), \quad (3.1)$$

with the supplemental Neumann boundary condition  $\partial_r \phi(r=0, t) = 0$ . The mode expansion of  $\phi$  can be written as

$$\phi(r, t) = i \int_{-\infty}^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} \left( a(k) e^{-i\vec{k} \cdot \vec{x}} - a^\dagger(k) e^{i\vec{k} \cdot \vec{x}} \right), \quad (3.2)$$

where  $\vec{k} \cdot \vec{x} = |k|t - kr$ . The Neumann boundary condition enforces  $a(k) = a(-k)$ , so that at  $r = 0$ ,  $\phi(r, t)$  can be expressed as an integral over  $k > 0$  modes.

In order to make the problem solvable, as we will see, we make an approximation that favors photons in the vicinity of  $\omega_0$ , so that  $\omega^{3/2}$  in (2.13) is replaced by  $\omega_0 \sqrt{\omega}$ . Then,

$$\int_0^\infty d\omega \omega^{3/2} (a(\omega) e^{-i\omega t} + a^\dagger(\omega) e^{i\omega t}) \approx \sqrt{\pi} \omega_0 \partial_t \phi(0, t). \quad (3.3)$$

If  $d$  is complex we absorb the phases of  $d, d^*$  into the definition of  $S^\pm$  without changing the commutation relations (2.14). Finally, we obtain the hamiltonian<sup>2</sup>

$$H = H_0^{\text{field}} + \frac{\omega_0}{2} S_3 + \frac{\beta}{4} \partial_t \phi(0, t) (S^+ + S^-), \quad (3.4)$$

where

$$\beta = \sqrt{\frac{32}{3}} d \omega_0. \quad (3.5)$$

To lowest order in perturbation theory, the hamiltonian (3.4) gives the well-known result  $\Gamma_{\text{decay}} = \frac{4}{3} d^2 \omega_0^3$ . Our solution below presents all-order  $\beta^2$  corrections to this result based on (3.4).

Though we need to make the approximation (3.3) in order to solve the model, fortunately in a one-dimensional fiber geometry, one finds  $\omega^{1/2}$  rather than  $\omega^{3/2}$  in (2.13),

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<sup>2</sup> This hamiltonian is closely related to the so-called Lee model and its generalizations[10], except that in our model the fermion fields are fixed at one spacial location. The latter two papers in [10] study the renormalization problems associated with what amounts to the rotating-wave approximation.

so that the model is exactly solvable without any further approximations[2]. All of the subsequent results of this paper hold in a fiber geometry with

$$\beta_{\text{fiber}} = \left( \frac{16\pi}{\mathcal{A}_{eff}} \right)^{1/2} d, \quad (3.6)$$

where  $\mathcal{A}_{eff}$  is the effective cross-sectional area of the fiber.

The parameter  $\beta^2/8\pi$  is a dimension-less coupling constant, where the strong coupling regime is large  $\beta^2$ . Using the lowest order perturbative result for  $\Gamma_{\text{decay}}$ , this coupling can also be expressed as

$$g \equiv \frac{\beta^2}{8\pi} \approx \frac{1}{\pi} \frac{\Gamma_{\text{decay}}}{\omega_0}, \quad (g \ll 1). \quad (3.7)$$

In order to get some idea of the size of the coupling for real atoms, we can consider a hydrogen-like atom with the nuclear charge  $Ze$ . The largest value of  $\beta$  occurs for the  $1s$  to  $2p$  transition, where  $g = Z^2 e^6 2^{11} 3^{-9} / \pi \approx Z^2 10^{-8}$ . Here,  $g$  goes as  $e^6$  since the Coulomb interaction determines both  $\omega_0$  and  $d$ . One can perhaps hope for larger values of  $g$  in artificial atoms, e.g. quantum dots, wherein  $d$  and  $\omega_0$  are fixed by distinct physics.

## 4. Mapping to the Kondo Model

### 4.1. The Exact Scattering Spectrum

In [2], a single atom in a fiber, which is described by a hamiltonian of the form (3.4) except that the theory lives on the full line  $-\infty < x < \infty$  and  $\beta$  is different, was mapped onto the Kondo model. Here, the result is even simpler since one doesn't have to fold the system. Namely,  $H$  is related by a unitary transformation  $U$  to the bosonized form of the anisotropic Kondo hamiltonian  $H_K$ :

$$H = U^\dagger H_K U, \quad (4.1)$$

where

$$H_K = H_0^{\text{field}} + \frac{\omega_0}{2} \left( S^+ e^{i\beta\phi(0)/2} + S^- e^{-i\beta\phi(0)/2} \right), \quad (4.2)$$

and

$$U = \frac{1}{\sqrt{2}} e^{i\beta S_3 \phi(0)/4} (S_3 + S_+ + S_-). \quad (4.3)$$

Since  $H$  and  $H_K$  are related by a unitary transformation, the quantum mechanics can be formulated in either the ‘optical picture’ or the ‘Kondo picture’. Matrix elements of operators  $\mathcal{O}$  in the two pictures are simply related:

$$\langle \psi' | \mathcal{O} | \psi \rangle_{\text{optical}} = {}_K \langle \psi' | \mathcal{O}_K | \psi \rangle_K, \quad (4.4)$$

where  $|\psi\rangle_K = U|\psi\rangle$ , and  $\mathcal{O}_K = U\mathcal{O}U^\dagger$ . We remark that the trivial atomic hamiltonian  $H_0^{\text{atom}}$  in the optical picture becomes the interaction in the Kondo picture.

Both models (3.4)(4.2) have ultra-violet divergences, and furthermore the quantum operator  $U$  is in need of regularization. We will take the point of view that the equation (4.1) defines the regularization of  $H$  once we have regularized  $H_K$  and  $U$ . The regularization of  $H_K$  is as in the sine-Gordon model and amounts to properly normal-ordering the exponential operators; this leads to the anomalous scaling dimension  $-g$  for the parameter  $\omega_0$ . It can be checked that this is consistent with the renormalization performed directly in the optical hamiltonian  $H$  as was done in [11], where it was shown that to lowest order the beta function reads  $\mu\partial_\mu\omega_0 = -g\omega_0$ .

The physical parameters of the optical problem are  $g$ , which is governed by the strength of the dipole coupling, and the energy scale set by the two-level splitting. The parameter  $\omega_0$  in (4.2) is a bare, unphysical parameter; physical energy scales are a function of  $\omega_0$ ,  $g$  and an ultraviolet cut-off  $\mu$ . In the scattering theory solution presented below, physical energy scales will be set by the parameter  $\omega_B$ . This scale is a known function which we will not need, but for completeness include from [2]:

$$\omega_B = \frac{1}{\sqrt{\pi}} \cot\left(\frac{\pi g}{2-2g}\right) \frac{\Gamma\left(\frac{1-2g}{2-2g}\right)}{\Gamma\left(\frac{2-3g}{2-2g}\right)} \left(\frac{\omega_0(\mu)}{2} \Gamma(1-g)\right)^{\frac{1}{1-g}}. \quad (4.5)$$

In the Kondo model the physical parameters are the dimension-less anisotropy parameter  $g$  and the ‘Kondo-temperature’  $T_K$ , which is related to  $\omega_B$  as follows:

$$T_K = \tan\left(\frac{\pi g}{2-2g}\right) \omega_B. \quad (4.6)$$

$T_K$  is defined such that the atomic impurity contribution to the partition function behaves as

$$Z_{\text{atom}} = 2 \cosh(T_K/T) + \mathcal{O}(g), \quad (4.7)$$

thus  $T_K$  is convenient for describing thermodynamical properties, as was done in [2]. For small  $g$ ,  $T_K \approx \omega_0/2$ ; this is consistent with the fact that when  $g = 0$ ,  $Z_{\text{atom}}$  should represent the partition function for a two-level system with energies  $\pm\omega_0/2$ . Henceforth, we will express all energies in terms of  $\omega_B$ .

Most importantly, the spectrum of  $H$  and  $H_K$  are identical. Thus we can infer what are the quantum states that exactly diagonalize the atom-field interaction from knowledge of the exact spectrum in the Kondo model<sup>3</sup>. This spectrum consists of a rich spectrum of massless particles which have a sine-Gordon-like character. The origin of the sine-Gordon spectrum is due to the fact that a bulk sine-Gordon interaction is compatible, as far as integrability goes, with the boundary interaction. (See below.) This is analogous to the treatment of the boundary sine-Gordon theory in [15], the difference being in the reflection S-matrices. Namely, the spectrum consists of a soliton and anti-soliton, and  $[1/g - 2]$  breathers which can be viewed as soliton-anti-soliton bound states. The meaning of this spectrum will be elaborated upon in section 5. The point  $g = 1/2$ , the so-called Toulouse point of Kondo physics, can be formulated as a free fermion theory using bosonization. (See below.) In the repulsive regime  $g \geq 1/2$ , there are only solitons and anti-solitons. For  $g > 1$ , the interaction is an irrelevant operator and the theory breaks down. In the bulk these particles propagate with a massless dispersion relation:

$$\begin{aligned} E_a &= P_a = \mu_a e^\theta && \text{for right - movers} \\ E_a &= -P_a = \mu_a e^{-\theta} && \text{for left - movers} \end{aligned} \tag{4.8}$$

where  $a \in \{1, 2, \dots, (1-g)/g, s, \bar{s}\}$  is an index running over the breathers and solitons, and  $\theta$  is a rapidity variable. The parameters  $\mu_a$  are related as follows:

$$\mu_n = 2 \mu \sin\left(\frac{n\pi g}{2-2g}\right), \quad \mu_s = \mu_{\bar{s}} = \mu, \tag{4.9}$$

where  $\mu$  is an arbitrary, unphysical energy scale which can be set to 1.

The interactions of these particles with the atom are encoded in the exact reflection S-matrices for these particles at the boundary[12][13]. Let  $R_a^b(\theta)$  denote the S-matrix for

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<sup>3</sup> We refer the reader to [12][13] for a description of the Kondo model that is the most useful in our context. An exact Bethe-ansatz solution of the Kondo model was found in [14]. References to other earlier papers can be found in [13].

particle of type  $a$  to scatter off the boundary and be reflected into a particle of type  $b$ . Defining  $\theta_B$  as:

$$\omega_B = \mu e^{\theta_B}, \quad (4.10)$$

the explicit expressions for the reflection S-matrices for the left-movers are

$$\begin{aligned} R_+^-(\theta) &= R_-^+(\theta) = \tanh\left(\frac{\theta + \theta_B}{2} - \frac{i\pi}{4}\right) \\ R_+^+(\theta) &= R_-^-(\theta) = 0 \\ R_n^n(\theta) &= \frac{\tanh\left(\frac{\theta + \theta_B}{2} - \frac{i\pi gn}{4(1-g)}\right)}{\tanh\left(\frac{\theta + \theta_B}{2} + \frac{i\pi gn}{4(1-g)}\right)}. \end{aligned} \quad (4.11)$$

Here  $+$  =  $s$  and  $-$  =  $\bar{s}$ .

It will be useful to express the reflection S-matrices in terms of resonances  $\widehat{\omega}_{0,n}$ , widths  $\Gamma_n$ , and physical energies  $E$ :

$$\begin{aligned} R_+^-(E) &= \frac{\widehat{\omega}_{0,s}^2 + E^2}{\widehat{\omega}_{0,s}^2 - E^2 + i\Gamma_s E} \\ R_n^n(E) &= \frac{\widehat{\omega}_{0,n}^2 - E^2 - i\Gamma_n E}{\widehat{\omega}_{0,n}^2 - E^2 + i\Gamma_n E}, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \widehat{\omega}_{0,s} &= \omega_B, \quad \Gamma_s = 2\omega_B \\ \widehat{\omega}_{0,n} &= 2 \sin\left(\frac{n\pi g}{2-2g}\right) \omega_B, \quad \Gamma_n = 4 \sin^2\left(\frac{n\pi g}{2-2g}\right) \omega_B. \end{aligned} \quad (4.13)$$

Near the resonances  $E \approx \widehat{\omega}_{0,n}$ , the reflection S-matrices have a Lorentzian signature:

$$R_n^n \approx -\frac{(\Gamma_n/2)^2}{(\widehat{\omega}_{0,n} - E)^2 + (\Gamma_n/2)^2}. \quad (4.14)$$

For the solitons, one has

$$R_+^- \approx -i \frac{(\Gamma_s/2)^2}{(\widehat{\omega}_{0,s} - E)^2 + (\Gamma_s/2)^2}. \quad (4.15)$$

As we will show in section 5, the quantities  $\widehat{\omega}_{0,n}, \Gamma_n$  determine the energy spectrum of eigenstates in finite volume.

It is well known in the quantum sine-Gordon theory literature that the  $n = 1$  breather is the particle corresponding to the scalar field  $\phi$  itself[16]. To make this more explicit, in

[2] electric field correlation functions were computed in the single 1-breather approximation which is remarkably good for values of  $g$  up to  $\sim 1/5$ . Because of (4.14), the vacuum power spectrum is Lorentzian where  $\hat{\omega}_{0,1}$  and  $\Gamma_1$  represent the Lamb-shifted two-level atomic energy splitting and decay rate. These agree with lowest order perturbative computations:

$$\begin{aligned}\hat{\omega}_{0,1} &= \omega_0 + \mathcal{O}(g) \\ \Gamma_1 &= g\pi\omega_0 + \mathcal{O}(g^2).\end{aligned}\tag{4.16}$$

## 5. Topological Charge

In the usual bulk sine-Gordon theory on the full line, the soliton and anti-soliton carry topological charge  $Q = \pm 1$ , where

$$Q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi = \frac{\beta}{2\pi} (\phi(\infty) - \phi(-\infty)).\tag{5.1}$$

In our boundary version of the problem, solitons are still characterized by a charge  $Q = \beta(\phi(r = \infty) - \phi(r = 0))/2\pi$ . The value of  $\phi$  at  $r = 0$  isn't fixed, thus topological charge conservation can be violated. For simplicity, set  $\phi(r = \infty) = 0$ . An incoming soliton ( $Q = 1$ ) with  $\phi(r = 0) = -2\pi/\beta$  can be reflected at  $r = 0$  into an outgoing anti-soliton with  $\phi(r = 0) = 2\pi/\beta$ , violating charge conservation by 2 units. In this process, the field at the boundary interpolates between  $-2\pi/\beta$  and  $2\pi/\beta$ . The fact that  $R_{\pm}^{\pm} = 0$  means that only processes which violate topological charge conservation by two units are allowed in the soliton sector.

We can relate the topological charge of solitons to simple properties of the electric field  $E \propto \partial_t \phi$ . Imagining the soliton as a pulse localized in space, then an incoming soliton at time  $-t$  and an outgoing anti-soliton at time  $t$ , where  $t = 0$  corresponds to complete absorption, have pulse profiles in space simply related by  $\phi \rightarrow -\phi$ . Thus, the phase of the electric field of the outgoing anti-soliton is shifted by  $\pi$  in comparison to the incoming anti-soliton.

Consider a soliton scattering process where the atom is initially in its ground state. When the soliton is absorbed, the atom is precisely in its excited state; it then emits an anti-soliton and returns to its ground state in the far future. This can be seen by examining the original optical states  $|\pm\rangle$  in the Kondo picture. Defining  $|\pm, \phi(0)\rangle_K = U|\pm\rangle$ , one has

$$|\pm, \phi(0)\rangle_K = \frac{1}{\sqrt{2}} \left( e^{i\beta\phi(0)/4} |+\rangle \pm e^{-i\beta\phi(0)/4} |-\rangle \right).\tag{5.2}$$

Note that

$$|-, \phi(0) + \frac{2\pi}{\beta}\rangle_K = i|+, \phi(0)\rangle_K, \quad |-, \phi(0) + \frac{4\pi}{\beta}\rangle_K = -|-, \phi(0)\rangle_K. \quad (5.3)$$

Thus, a soliton incident on the atom in its ground state in the far past corresponds to  $|-, -2\pi/\beta\rangle$ , and when the soliton is completely absorbed  $\phi(0) = 0$  and the state has evolved to  $i|+, 0\rangle$ , i.e. is in its excited state. In the far future,  $\phi(0) = 2\pi/\beta$ , and the state is  $-|-, 2\pi/\beta\rangle$ . The atom has necessarily returned to its ground state due to energy conservation. Note however that the atomic state has changed phase by  $\pi$ .

The fact that soliton absorption excites that atom to precisely its excited state indicates that the solitons are the most fundamental excitations. As we will see below, at strong coupling only the solitons remain in the spectrum.

## 6. Optical Phonon Spectrum and the Jaynes-Cummings Model Limit

Though the scattering theory described above provides an exact solution of the model, its precise meaning is not entirely clear from what we have done so far. The original Hilbert space,  $\mathcal{H}$ , consisting of the two-level space tensored with the photon Hilbert space, has been replaced with an infrared description  $\mathcal{H}_{\text{particle}}$ . The latter Hilbert space contains a spectrum of particles with continuous energies, and does not have a two-level structure. The distinguishing feature of this spectrum of particles is that they provide a basis that diagonalizes the atom-field interaction; the existence of the two-level atomic Hilbert space is encoded in the reflection S-matrices  $R$ . A similar treatment of the simpler case of harmonic-oscillator atomic impurities was given in [17]. Irrespective of these remarks, the free field hamiltonian  $H_0^{\text{field}}$  is a linear theory in vacuum, so an intrinsic understanding of the non-linear sine-Gordon-like spectrum seems called for.

It is also desirable to describe eigenstates of the atom-field interaction in a fashion that is more understandable in terms of the original Hilbert space  $\mathcal{H}$ . This is a difficult problem in general, since it requires computing the inner-product of states in  $\mathcal{H}$  with states in  $\mathcal{H}_{\text{particle}}$ . Exact eigenstates can be described in the space  $\mathcal{H}$  for the much simpler Jaynes-Cummings (JC) model. In this section we describe how information about the atom-field eigenstates is encoded in the scattering theory description, and indeed we show how to obtain the known JC eigen-energies in the appropriate limit. This exercise leads to the understanding that for each particle of the spectrum of section 3 there is an associated optical phonon, in a sense that will be explained.

### 6.1. Jaynes-Cummings Model

The Jaynes-Cummings model[18] is an exactly solvable model of the atom-field interaction which arises under various approximations to the hamiltonian (3.4). Let us suppose that the one-dimensional volume of the cavity is  $L$ . The first approximation is to only consider a single discrete photon mode of the cavity of frequency  $\omega_c$  near  $\omega_0$ . Namely, the free photon Hilbert space is built from the single mode creation-annihilation operators  $a \equiv a(\omega_c)$  and  $a^\dagger = a^\dagger(\omega_c)$  satisfying  $[a, a^\dagger] = 1$ . Next, we make the rotating-wave approximation and drop the terms  $aS^-$  and  $a^\dagger S^+$  in (3.4). Finally, we drop the exponentials  $\exp(\omega_c - \omega_0)$  in the interaction. Recalling that in finite volume  $\int dk/\sqrt{2\pi} \rightarrow \frac{1}{\sqrt{L}} \sum_k$ , one obtains the hamiltonian:

$$H = \omega_c a^\dagger a + \frac{\omega_0}{2} S_3 + \alpha (aS^+ + a^\dagger S^-), \quad (6.1)$$

where

$$\alpha = \sqrt{\frac{\pi g \omega_c}{L}}. \quad (6.2)$$

The volume  $L$  drops out of many physical quantities; for instance  $\Gamma_{\text{decay}} = \alpha^2 L|_{\omega_c=\omega_0} = \beta^2 \omega_0/8$ . In the limiting case of a micro-cavity of size tuned to the resonant frequency  $\omega_c = \omega_0$  with  $\omega_0/\pi = 1/L$ , one has  $\alpha_{\text{micro}} = \sqrt{g\pi} \omega_0$ .

The above hamiltonian is easily diagonalized exactly. Let  $|N, \pm\rangle$  denote a basis of states with  $N$  photons and  $\pm$  the two-level states of the atom:

$$a^\dagger a |N, \pm\rangle = N |N, \pm\rangle, \quad S_3 |N, \pm\rangle = \pm |N, \pm\rangle. \quad (6.3)$$

The hamiltonian has a  $2 \times 2$  block diagonal form where the blocks act on the pairs of states  $|N-1, +\rangle, |N, -\rangle$ . Each block is easily diagonalized leading to an energy spectrum  $\mathcal{E}_N^\pm$ ,  $N = 1, 2, \dots$ :

$$\mathcal{E}_N^\pm = \omega_c(N-1/2) \pm \frac{1}{2} \sqrt{(\omega_c - \omega_0)^2 + 4\alpha^2 N}, \quad (6.4)$$

where the eigenstates  $H\Psi_N^\pm = \mathcal{E}_N^\pm \Psi_N^\pm$  are given by

$$\begin{aligned} \Psi_N^+ &= \cos \Theta |N-1, +\rangle + \sin \Theta |N, -\rangle \\ \Psi_N^- &= -\sin \Theta |N-1, +\rangle + \cos \Theta |N, -\rangle, \end{aligned} \quad (6.5)$$

where

$$\tan \Theta = \frac{\sqrt{(\omega_c - \omega_0)^2 + 4\alpha^2 N} + (\omega_c - \omega_0)}{\sqrt{(\omega_c - \omega_0)^2 + 4\alpha^2 N} - (\omega_c - \omega_0)}. \quad (6.6)$$

This spectrum is observable experimentally[19].

## 6.2. Optical Phonon Spectrum

Let us describe how to obtain the JC spectrum from our scattering theory under certain limits. In order to compare with the single cavity mode approximation, we must place the atom inside a finite volume, letting the spherical cavity have a diameter  $L$ . There is now a quantization condition on allowed momenta which depends on the reflection S-matrices. Consider a wave-function for a particle from the spectrum of section 3 as it traverses the radius of the cavity, is reflected and returns to the edge. This wave-function picks up a phase which must equal unity:

$$e^{iPL} R(E) = 1. \quad (6.7)$$

Taking the logarithm, and using  $P = -E$ , one obtains

$$E + \frac{i}{L} \log R(E) = \frac{2\pi m}{L}, \quad (6.8)$$

where  $m$  is some integer. The quantity on the RHS determines the mode of the cavity:  $\omega_c = 2\pi m/L$ .

Let us now make two approximations. As in the JC model we assume the cavity is tuned near resonance  $\omega_c \approx \hat{\omega}_{0,n}$  for the quantization of the  $n$ -th breather with  $R = R_n^n$  in (6.8). Secondly, at weak coupling, for the low  $n$  breathers, it is safe to suppose  $\Gamma_n \ll |\hat{\omega}_{0,n} - E|$ , since  $\Gamma_n \approx n^2 \pi^2 g^2 \omega_B$  and  $\hat{\omega}_{0,n} \approx n \pi g \omega_B$ . Under these approximations,  $\log R_n^n \approx -i\Gamma_n/(\hat{\omega}_{0,n} - E)$ . Inserting this into (6.8), one obtains a quadratic equation for  $E$ , the solution for the  $n$ -th breather being:

$$E_{N=1,n}^{\pm}(\omega_c) = \frac{\hat{\omega}_{0,n} + \omega_c}{2} \pm \frac{1}{2} \left( (\omega_c - \hat{\omega}_{0,n})^2 + 4\Gamma_n/L \right)^{1/2}. \quad (6.9)$$

Multiparticle ( $N > 1$ ) eigenstates are characterized by more complicated quantization conditions involving the bulk S-matrix[15]. Since the ground state energy of the JC model is  $-\omega_0/2$ , whereas that of the scattering description is set to zero, one should subtract  $\omega_0/2$  from the above energies in order to compare. Using the small  $g$  limits,

$$E_{N=1,n}^{\pm}(\omega_c) - \omega_0/2 \approx \frac{(n-1)}{2} \omega_0 + \omega_c \pm \frac{1}{2} \left( (\omega_c - n\omega_0)^2 + 4n^2 g \pi \omega_0 / L \right)^{1/2}. \quad (6.10)$$

From (6.2), with  $\omega_c \approx \omega_0$ , one sees that for the  $n = 1$  breather, one-particle energies correspond precisely to the one-photon JC energies:

$$E_{N=1,n=1}^{\pm} - \omega_0/2 = \mathcal{E}_{N=1}^{\pm}. \quad (6.11)$$

At  $\omega_c = \omega_0$ , the  $N = 1$  JC eigenstates are symmetrically split about  $\omega_0/2$ . Let us define then a scaled energy splitting  $E'$  such that:

$$\frac{\mathcal{E}_1^\pm - \omega_0/2}{\omega_0} = \pm E' = \pm(g/2)^{1/2}. \quad (6.12)$$

We can then compare this with the ‘exact’ prediction (6.8). Setting  $E' = (E - \hat{\omega}_{0,1})/\hat{\omega}_{0,1}$  in (6.8) where  $R$  is the reflection S-matrix for the first breather  $R_1^1$ , and  $\omega_c = \hat{\omega}_{0,1} = 2\pi/L$ , one finds that  $E'$  is a solution of the equation

$$E' + \frac{i}{2\pi} \log \left( \frac{E'^2 + 2E' + 2i \sin(\pi g/(2 - 2g))(E' + 1)}{E'^2 + 2E' - 2i \sin(\pi g/(2 - 2g))(E' + 1)} \right) = 0. \quad (6.13)$$

The differences between the JC model and the result (6.13) are displayed in figure 1. At small  $g$  they of course agree very well. The strong departure from the JC model at  $g = 1/2$  is due to the fact that this is the threshold for the disappearance of this particle.

The higher  $n$ -breather  $N = 1$  particle states are not predicted by the JC model. We believe this is because higher resonances  $E \approx n\omega_0$  are eliminated in the rotating wave approximation. Based on the above observations we can formulate the following appealing picture of the spectrum. At large volumes, the energies behave as

$$\begin{aligned} E_{N=1,n}^+ &\approx \omega_c, & E_{N=1,n}^- &\approx \hat{\omega}_{0,n}, & \omega_c &\gg \hat{\omega}_{0,n}, \\ E_{N=1,n}^+ &\approx \hat{\omega}_{0,n}, & E_{N=1,n}^- &\approx \omega_c, & \omega_c &\ll \hat{\omega}_{0,n}. \end{aligned} \quad (6.14)$$

Thus the  $E^+$  branch is optical phonon-like at small  $\omega_c$  and photon-like at large  $\omega_c$ , whereas for  $E^-$  this is reversed. For  $n = 1$ , these states are the precursor to the well-known polariton branches which occur in the context of a dielectric medium of atoms. (See for instance [20] and the next section.) Here, there is no medium, however these ‘vacuum-polaritons’ are characterized by the regimes where they are phonon-like, so we will simply refer to them as optical phonons. The  $n = 1$  breather optical phonon is the basic excitation of the JC model. Since the  $n$ -th breather leads to an optical phonon at energy approximately  $n\omega_0$  for small  $g$ , the latter can be understood as a bound state of  $n$  optical phonons of energy  $\omega_0$ .

We thus propose that our model contains a rich spectrum of optical phonons of energy  $\hat{\omega}_{0,n}$ ,  $\hat{\omega}_{0,s}$ , in correspondence with the breather/soliton spectrum in section 3. The  $n$ -th optical phonon is a bound state of  $n$  fundamental  $n = 1$  phonons. As  $g$  increases the higher optical phonons become unbound and disappear from the spectrum one by one. When  $g = 1/(n + 1)$ , from (4.13) one sees that  $\hat{\omega}_{0,n} = 2\hat{\omega}_{0,s}$ . Thus as  $g$  increases and

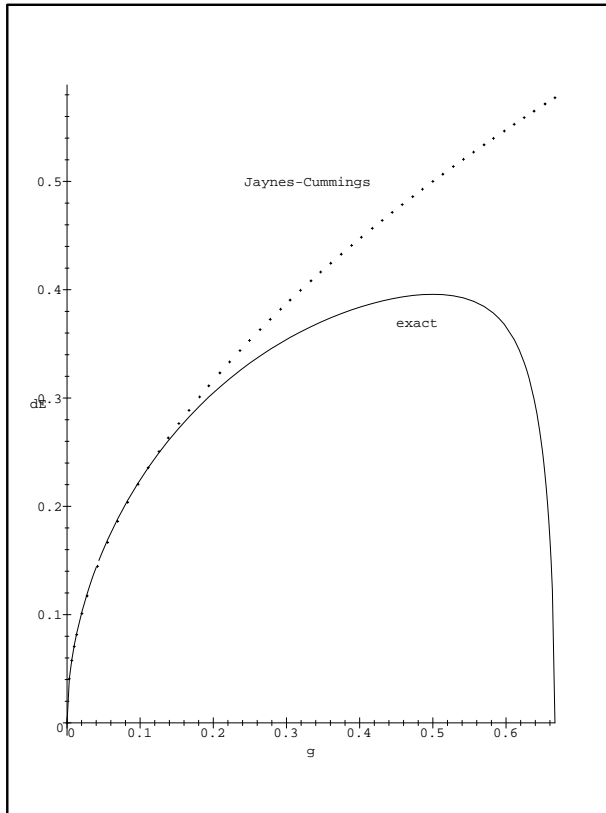


Figure 1. Comparison of the energy splitting of the  $N = 1$  photon Jaynes-Cummings eigenstates and the predictions of our model. The y axis is  $E'$ , which, for the JC model is defined in (6.12), whereas for our model it is a solution to (6.13).

reaches the point  $1/(n+1)$ , the  $n$ -th optical phonon unbinds into two solitonic phonons. This also implies that all of the  $n$ -th breather optical phonons can also be viewed as a bound state of two solitonic optical phonons. Finally, when  $g = 1/2$ , the  $n = 1$  optical phonon, which is the fundamental excitation of the JC model, also disappears from the spectrum leaving only the solitonic phonons for all  $g \geq 1/2$ .

## 7. Atomic Impurity in a Photonic Crystal

In this section we place the atom in a dielectric medium. One can generally model

such a medium with a dielectric constant

$$\varepsilon(\omega) = \varepsilon_\infty + (\varepsilon_\infty - \varepsilon_0) \frac{\omega_T^2}{\omega^2 - \omega_T^2}, \quad (7.1)$$

giving the dispersion relation  $\varepsilon(\omega) = k^2/\omega^2$ . (See for instance [20].) For simplicity, let us set  $\varepsilon_\infty = 1$ . The dispersion relation has two polariton branches  $\omega_\pm(k)$  as shown in figure 2. Since  $\omega_+(k=0) = \sqrt{\varepsilon_0}\omega_T$ , and  $\omega_-(k=\infty) = \omega_T$ , there is a gap between the two branches  $E_{gap} = (\sqrt{\varepsilon_0} - 1)\omega_T$ . Though this is not the manner in which photon bandgaps are thought to arise in e.g. [4], we believe it serves as a good model for the physics we are trying to study.

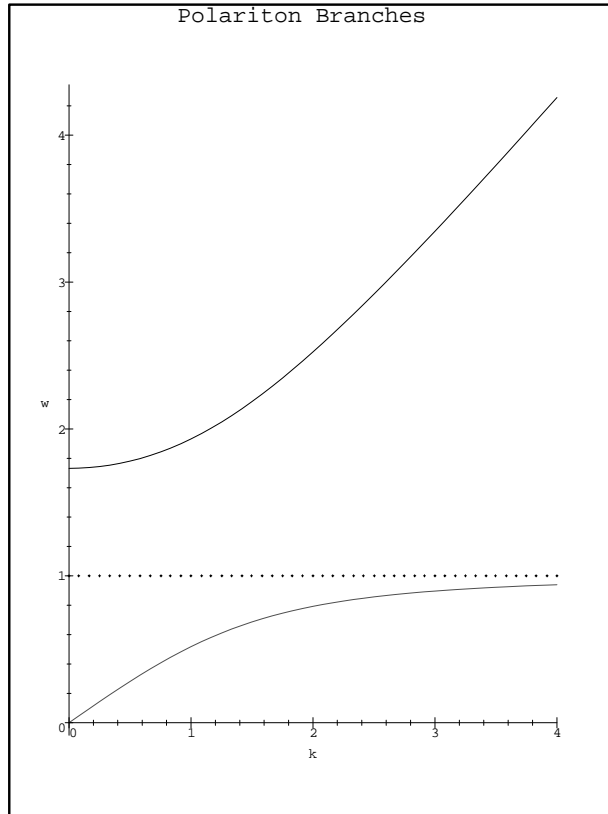


Figure 2. The two polariton branches of the dispersion relation following from (7.1), with  $\varepsilon_0 = 3$ . Both axes are in units of  $\omega_T$ . The upper branch intersects the  $y$ -axis at  $\sqrt{\varepsilon_0}$ .

In a certain limit, the dispersion relation is approximately relativistic. Namely, let us treat the lower branch as constant  $\omega_-(k) = \omega_T$ , and shift the energy of the upper branch  $\omega_+ = \omega_T + \omega'$ . Then, if  $\varepsilon_0$  is large,  $\omega'$  is large compared to  $\omega_T$  and  $\omega'^2 - k^2 \approx m_{\text{gap}}^2$  with

$$m_{\text{gap}} = \sqrt{\varepsilon_0 - 1} \omega_T. \quad (7.2)$$

For  $\varepsilon_0 \gg 1$ ,  $m_{\text{gap}} \approx E_{\text{gap}}$ . Alternatively, one can view this limit as taking  $\omega_T \rightarrow 0$ , and  $\varepsilon_0 \rightarrow \infty$ , keeping  $m_{\text{gap}} = \sqrt{\varepsilon_0} \omega_T$  fixed. In this limit the lower branch disappears and the upper branch is a massive relativistic dispersion relation.

One can model this gap in the medium while preserving the integrability by adding a sine-Gordon interaction in the bulk:

$$H_0^{\text{field}} \rightarrow H_0^{\text{field}} - \frac{m^2}{\beta^2} \int_0^\infty dr \cos(\beta\phi). \quad (7.3)$$

Choosing  $m$  as in (7.2), the above interaction gives the photon the proper dispersion relation. Mapping onto the Kondo model as in (4.1), one now obtains

$$H_K^{(m)} = \left[ \int_0^\infty dr \left( \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_r \phi)^2 - \frac{m_{\text{gap}}^2}{\beta^2} \cos(\beta\phi) \right) \right] + \frac{\omega_0}{2} \left( S^+ e^{i\beta\phi(0)/2} + S^- e^{-i\beta\phi(0)/2} \right). \quad (7.4)$$

The desirable feature of the theory (7.4) is that the bulk and boundary interactions are compatible as far as integrability is concerned. Namely, the exact bulk sine-Gordon spectrum also diagonalizes the boundary interaction with the atom. The spectrum of the model is the same as in section 3, except that the particles are now massive with a mass scale determined by  $m_{\text{gap}}$ . The dispersion relation is now

$$E_a = M_a \cosh \theta, \quad P_a = M_a \sinh \theta, \quad (7.5)$$

satisfying  $E_a^2 - P_a^2 = M_a^2$ . The masses  $M_a$  are related as in (4.9),

$$M_n = 2M_s \sin \frac{n\pi g}{2-2g}, \quad (7.6)$$

but now the soliton mass  $M_s$  is a physical parameter depending on  $m_{\text{gap}}$ . Since the first breather is identified with the photon, we will set

$$M_1 = m_{\text{gap}}. \quad (7.7)$$

In the literature on the Kondo model, a bulk mass term as in (7.4) is never considered since one deals with massless fermions near the Fermi surface. We believe the model (7.3) is integrable since the bulk massive spectrum is compatible with the spectrum that diagonalizes the boundary interaction. The boundary reflection S-matrices for the model (7.4) are unknown. For simplicity, in this paper we will limit ourselves to the model at  $g = 1/2$  and derive the S-matrices in the next section. We intend to present the general case elsewhere.

## 8. Free Fermion Point: Massive Case

Remarkably, at the point  $g = 1/2$  in the strong coupling regime, the theory is equivalent to a free fermion theory. Though this is well-known in the Kondo physics literature, this is a novel phenomenon in the optical context since at this value of the coupling the perturbatively dressed photon, in the sense of the Jaynes-Cummings model, disappears entirely from the spectrum. The theory at this coupling will be considered in greater detail in [21]. Here, we consider the bulk massive case, which has not been studied before, with applications to photonic crystals in mind.

### 8.1. Physical Parameters

We first define the physical parameters. When  $g = 1/2$ ,  $T_K$  diverges however  $\omega_B$  remains finite<sup>4</sup>. For  $g = 1/2$  we define the physical parameters as

$$\hat{\omega}_0 = 2\hat{\omega}_{0,s} = 2\omega_B, \quad \Gamma_{\text{decay}} = 2\hat{\omega}_0. \quad (8.1)$$

Note that this precisely corresponds to the 1-breather expressions  $\hat{\omega}_{0,1}$  and  $\Gamma_1$  evaluated at  $g = 1/2$ . The parameter  $\hat{\omega}_0$  represents the energy splitting of the atom; note that since this is twice  $\hat{\omega}_{0,s}$ , the soliton is in resonance with *half* of the energy splitting.

From (7.6) one sees that the breather mass  $M_1$  is exactly twice the soliton, or fermion, mass  $M_s$ , which reflects the fact that the lowest breather is a soliton/anti-soliton bound state and just becomes unbound at  $g = 1/2$ . In this section, we will express everything in terms of the fermion mass  $M \equiv M_s = m_{\text{gap}}/2$ .

### 8.2. Reflection S-matrices

We now derive the reflection S-matrices in a treatment that is similar to what was done for the boundary sine-Gordon theory in [22].

In the massless limit, the scalar field decomposes into left and right-moving components  $\phi = \varphi(z^+) + \bar{\varphi}(z^-)$ , where  $z^\pm = t \pm r$ . The Neumann boundary condition  $\partial_r \phi = 0$  then reads  $\partial_{z^+} \varphi(z^+) = \partial_{z^-} \bar{\varphi}(z^-)$  at  $r = 0$ . This implies

$$\varphi(z^+) = \bar{\varphi}(z^-) - \frac{\sigma}{\sqrt{4\pi}}, \quad (8.2)$$

where  $\sigma$  is some constant.

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<sup>4</sup> One finds  $\omega_B \propto \omega_0^2$ , which is consistent with the anomalous  $\sqrt{\text{mass}}$  dimension of  $\omega_0$ .

The fermion fields, with topological charge  $\pm 1$ , are the following:

$$\psi_{\pm} = e^{\pm i\sqrt{4\pi}} \varphi, \quad \bar{\psi}_{\pm} = e^{\mp i\sqrt{4\pi}} \bar{\varphi}. \quad (8.3)$$

In terms of the fermions, the Neumann boundary condition reads

$$\psi_{\pm} = e^{\mp i\sigma} \bar{\psi}_{\mp}, \quad (r = 0) \quad (8.4)$$

which breaks the topological charge symmetry. The action which enforces the Neumann boundary condition is

$$\begin{aligned} S_{\text{free}} = \int_0^{\infty} dr dt & \left[ i\psi_{-}(\partial_t - \partial_r)\psi_{+} + i\bar{\psi}_{-}(\partial_t - \partial_r)\bar{\psi}_{+} - M(\bar{\psi}_{-}\psi_{+} + \psi_{-}\bar{\psi}_{+}) \right] \\ & - i \int dt (e^{i\sigma}\psi_{+}\bar{\psi}_{+} + e^{-i\sigma}\psi_{-}\bar{\psi}_{-}). \end{aligned} \quad (8.5)$$

Using (8.2), the interaction can be written as

$$S_{\text{int}} = -\frac{\lambda}{2} \int dt \left[ (\psi_{+}a_{-} + \bar{\psi}_{-}a_{+})S_{+} + S_{-}(a_{+}\psi_{-} + a_{-}\bar{\psi}_{+}) \right], \quad (8.6)$$

where  $a_{\pm} = e^{\mp i\sigma/2}$ . The parameter  $\lambda$ , with units of  $\sqrt{\text{mass}}$ , will be related to  $\omega_B$  below. The  $a_{\pm}$  should be regarded as fermionic operators:

$$a_{\pm}\psi = -\psi a_{\pm}, \quad (8.7)$$

for  $\psi \in (\psi_{\pm}, \bar{\psi}_{\pm})$ . The complete action is  $S = S_{\text{free}} + S_{\text{int}}$ .

The boundary equations of motion which follow from varying the action with respect to the fermion fields are, at  $r = 0$ :

$$\begin{aligned} \psi_{+} - e^{-i\sigma}\bar{\psi}_{-} &= \frac{i\lambda}{2}S_{-}a_{+}, \\ \bar{\psi}_{+} - e^{-i\sigma}\psi_{-} &= \frac{i\lambda}{2}a_{+}S_{+}, \end{aligned} \quad (8.8)$$

and the hermitian conjugate of these. The time derivative of the operators on the RHS of (8.8) are determined by their commutation relations with the hamiltonian,  $\partial_t \mathcal{O} = -i[\mathcal{O}, H]$ . Using (8.7) and  $\{S_{+}, S_{-}\} = 1$ , one obtains

$$\begin{aligned} \partial_t(S_{-}a_{+}) &= \frac{i\lambda}{2}(\psi_{+} + a_{+}^2 \bar{\psi}_{-}) \\ \partial_t(a_{+}S_{+}) &= -\frac{i\lambda}{2}(\bar{\psi}_{+} + a_{+}^2 \psi_{-}) \end{aligned} \quad (8.9)$$

and their hermitian conjugates (at  $r = 0$ ). Combining (8.8), (8.9) and their hermitian conjugates, one obtains two independent equations at  $r = 0$ :

$$\begin{aligned}\partial_t (\psi_+ - e^{-i\sigma} \bar{\psi}_-) &= -\frac{\lambda^2}{4} (\psi_+ + e^{-i\sigma} \bar{\psi}_-) \\ \partial_t (\bar{\psi}_+ - e^{-i\sigma} \psi_-) &= \frac{\lambda^2}{4} (\bar{\psi}_+ + e^{-i\sigma} \psi_-).\end{aligned}\tag{8.10}$$

The reflection S-matrices can be easily computed from (8.10). The fermion fields have the mode expansions

$$\begin{aligned}\psi_+(r, t) &= \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta e^{-\theta/2} \left( A_-(\theta) e^{-ik \cdot r} - A_+^\dagger(\theta) e^{ik \cdot r} \right) \\ \bar{\psi}_+(r, t) &= \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta e^{+\theta/2} \left( A_-(\theta) e^{-ik \cdot r} + A_+^\dagger(\theta) e^{ik \cdot r} \right),\end{aligned}\tag{8.11}$$

with  $\psi_- = \psi_+^\dagger$ ,  $\bar{\psi}_- = \bar{\psi}_+^\dagger$ . The variable  $\theta$  is the rapidity:  $k \cdot r = M \cosh(\theta)t - M \sinh(\theta)r$ , and the  $A, A^\dagger$  are fermion annihilation/creation operators satisfying  $\{A_\pm(\theta), A_\pm^\dagger(\theta')\} = \delta(\theta - \theta')$ .

The scattering of the particles off the boundary at  $r = 0$  can be formulated by formally introducing a boundary operator  $\mathcal{B}$  satisfying the algebraic relation[23]

$$A_a^\dagger(\theta) \mathcal{B} = R_a^b(\theta) A_b^\dagger(-\theta) \mathcal{B}.\tag{8.12}$$

Separating the integral over  $\theta$  in (8.11) into  $\theta < 0$  and  $\theta > 0$ , and making a change of variables  $\theta \rightarrow -\theta$  in the former, one obtains

$$\begin{aligned}& \left[ -e^{-\frac{\theta}{2}} \left( iM \cosh \theta + \frac{\lambda^2}{4} \right) A_+^\dagger(\theta) - e^{-i\sigma} e^{\frac{\theta}{2}} \left( iM \cosh \theta - \frac{\lambda^2}{4} \right) A_-^\dagger(\theta) \right] \mathcal{B} \\ &= \left[ e^{\frac{\theta}{2}} \left( iM \cosh \theta + \frac{\lambda^2}{4} \right) A_+^\dagger(-\theta) + e^{-i\sigma} e^{-\frac{\theta}{2}} \left( iM \cosh \theta - \frac{\lambda^2}{4} \right) A_-^\dagger(-\theta) \right] \mathcal{B} \\ & \left[ e^{\frac{\theta}{2}} \left( iM \cosh \theta - \frac{\lambda^2}{4} \right) A_+^\dagger(\theta) - e^{-i\sigma} e^{-\frac{\theta}{2}} \left( iM \cosh \theta + \frac{\lambda^2}{4} \right) A_-^\dagger(\theta) \right] \mathcal{B} \\ &= \left[ -e^{-\frac{\theta}{2}} \left( iM \cosh \theta - \frac{\lambda^2}{4} \right) A_+^\dagger(-\theta) + e^{-i\sigma} e^{\frac{\theta}{2}} \left( iM \cosh \theta + \frac{\lambda^2}{4} \right) A_-^\dagger(-\theta) \right] \mathcal{B}.\end{aligned}\tag{8.13}$$

From this one can read off the reflection S-matrices:

$$\begin{aligned}R_+^-(\theta) &= e^{-i\sigma} \frac{\sinh \theta}{\cosh(\theta - \gamma(\theta))}, & R_-^+(\theta) &= -e^{i\sigma} \frac{\sinh \theta}{\cosh(\theta - \gamma(\theta))} \\ R_+^+(\theta) &= R_-^-(\theta) = -\frac{\cosh \gamma(\theta)}{\cosh(\theta - \gamma(\theta))},\end{aligned}\tag{8.14}$$

where

$$e^{\gamma(\theta)} = \frac{\cosh \theta - i\Delta}{\cosh \theta + i\Delta}, \quad \Delta = \frac{\lambda^2}{2m_{\text{gap}}}. \quad (8.15)$$

The parameter  $\sigma$  is unphysical since it simply corresponds to a phase, hence we set it to zero.

One can easily check that  $R$  satisfies the unitarity and crossing symmetry constraints[23]:

$$\begin{aligned} R_a^b(\theta) R_b^c(-\theta) &= \delta_a^c \\ R_a^b\left(\frac{i\pi}{2} - \theta\right) &= -R_{\bar{a}}^{\bar{b}}\left(\frac{i\pi}{2} + \theta\right), \end{aligned} \quad (8.16)$$

where  $\bar{a} = -a$ .

One can also confirm that these reflection matrices have the proper massless limit. For the left-movers, the massless limit is obtained by letting  $\theta \rightarrow \theta - \alpha$ , and taking  $\alpha \rightarrow \infty$ ,  $M \rightarrow 0$  while keeping  $Me^\alpha/2 = \mu$  held fixed. In this way, the massless dispersion relation (4.8) is recovered from the massive one (7.5). Taking this limit in  $R$  one finds

$$\begin{aligned} R_{\pm}^{\mp} &\rightarrow \pm \tanh\left(\frac{\theta + \theta_B}{2} - \frac{i\pi}{4}\right), \\ R_{\pm}^{\pm} &\rightarrow 0, \end{aligned} \quad (8.17)$$

with

$$\omega_B = \mu e^{\theta_B} = \frac{\lambda^2}{4}. \quad (8.18)$$

This agrees with (4.11) up to unphysical phases. The relation (8.18) can now be used to express everything in terms of physical parameters. In particular, (8.18) combined with (8.1) gives

$$\Delta = \frac{\hat{\omega}_0}{m_{\text{gap}}}. \quad (8.19)$$

In terms of the physical energy  $E$ ,

$$\begin{aligned} R_+^-(E) &= \frac{\sqrt{E^2 - m_{\text{gap}}^2/4}}{E} \left( \frac{E^2 + \hat{\omega}_0^2/4}{E^2 - \hat{\omega}_0^2/4 + i\hat{\omega}_0\sqrt{E^2 - m_{\text{gap}}^2/4}} \right) \\ R_+^+ &= -\frac{m_{\text{gap}}}{2E} \left( \frac{E^2 - \hat{\omega}_0^2/4}{E^2 - \hat{\omega}_0^2/4 + i\hat{\omega}_0\sqrt{E^2 - m_{\text{gap}}^2/4}} \right). \end{aligned} \quad (8.20)$$

## 9. The Binding of Light to Atoms

Using results from the last section, we now study the binding of light to atoms in photon bandgap materials in the simplest case of the free fermion point. Recall the physical parameters are the renormalized two-level splitting  $\hat{\omega}_0$ , and the bandgap of the medium  $m_{\text{gap}}$ .

Light localized at, or bound to, the atom corresponds to a ‘boundary bound state’ of the kind described in [23]. Boundary bound states exist if there are poles in the reflection S-matrices  $R_a^b(\theta)$  in (8.14) on the physical strip  $0 \leq \text{Im } \theta \leq \pi$ . Poles in  $R$  occur when  $\cosh(\theta - \gamma(\theta)) = 0$ . Letting  $\theta = iu$ , poles occur at solutions of the equation

$$\sin^2 u + 2\Delta \sin u + \Delta^2 - 1 = 0. \quad (9.1)$$

Requiring the pole to be on the physical strip leads to a single boundary bound state at  $u = u_b$  satisfying

$$\sin u_b = 1 - \Delta \geq 0. \quad (9.2)$$

Thus there exists a light-atom bound state if  $\Delta \leq 1$ , or

$$\hat{\omega}_0 \leq m_{\text{gap}}. \quad (9.3)$$

The necessary condition (9.3) for the existence of a bound state is simple to understand: the atom normally decays spontaneously by emitting photons of energy  $\hat{\omega}_0$ , however such a photon cannot propagate in the medium and thus becomes bound to the atom.

The energy of the boundary bound state above the ground state energy is  $M \cos u_b$ , and this represents the binding energy:

$$E_{\text{bind}} = \left( \frac{m_{\text{gap}} \hat{\omega}_0}{2} \left( 1 - \frac{\hat{\omega}_0}{2m_{\text{gap}}} \right) \right)^{1/2}. \quad (9.4)$$

The bound state is stable if the binding energy is less than the mass of the lightest particle,  $E_{\text{bind}} < M$ , or  $\hat{\omega}_0 < m_{\text{gap}}/2$ .

The existence of this bound state can be used to design a filter for light near the binding energy. Recall that with no atomic impurities, light with energy  $E < m_{\text{gap}}$  is forbidden to propagate in the medium. The bound state is at an energy  $E_b < m_{\text{gap}}/\sqrt{2}$  due to the inequality (9.3), thus it is within the gap. If there are many atomic impurities, one can imagine that light of energy near  $E_b$  can resonate with the bound state and in

principle still propagate by tunneling along the impurities. One indication of this is the enhanced reflection amplitude for energies near  $E_b$ :

$$R_+^-(E) \approx \frac{i}{E - E_{\text{bind}}} \frac{(m_{\text{gap}} - \hat{\omega}_0)^2 \hat{\omega}_0}{4E_b^2}. \quad (9.5)$$

One can determine the reflection S-matrices for particle scattering off of the bound state itself[23]. In this simple case they turn out to be identical to the reflection S-matrices  $R_a^b$  for the scattering off the ground state computed in section 4.

## 10. Conclusions

We have found an exact solution to the problem of eigenstates of the atom-field interaction in a spherical cavity which shows a rich spectrum in comparison with models based on the rotating wave approximation. The spectrum has solitons and bound states, in spite of the fact that there is no medium, i.e. we are dealing with a single atom in vacuum. In a sense our work shows that the well-known solitons of classical non-linear optics have a counterpart for a single atom in vacuum.

By introducing a bulk mass term into the Kondo model in a way that preserves integrability we have shown how one can study analytically the occurrence and properties of photon-atom bound states in photonic crystals.

It is interesting to explore the possibility of utilizing the full spectrum of eigenstates in novel device applications. The solitonic mode being in a sense the most fundamental, one can imagine a ‘soliton laser’ based on the existence of this mode.

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